Note on the complexity of deciding the rainbow connectedness for bipartite graphs[*]

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Abstract

A path in an edge-colored graph is said to be a rainbow path if no two edges on the path have the same color. An edge-colored graph is (strongly) rainbow connected if there exists a rainbow (geodesic) path between every pair of vertices. The (strong) rainbow connection number of G, denoted by ($scr(G)$, respectively) $rc(G)$, is the smallest number of colors that are needed in order to make G (strongly) rainbow connected. Though for a general graph G it is NP-Complete to decide whether $rc(G) = 2$, in this paper, we show that the problem becomes easy when G is a bipartite graph. Moreover, it is known that deciding whether a given edge-colored (with an unbound number of colors) graph is rainbow connected is NP-Complete. We will prove that it is still NP-Complete even when the edge-colored graph is bipartite. We also show that a few NP-hard problems on rainbow connection are indeed NP-Complete.

Keywords: rainbow connection number; strong rainbow connection number; bipartite graph; NP-Complete; polynomial-time

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1   Introduction

We follow the terminology and notations of [2] and all graphs considered here are always finite and simple.

Let G be a nontrivial connected graph on which is defined a coloring c: $E(G) \rightarrow \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$, of the edges of G, where adjacent edges may be colored the same. A $u-v$ path P in G is a *rainbow path* if no two edges of P are colored the same. The graph G is *rainbow connected (with respect to c)* if G contains a rainbow $u-v$ path for every two vertices u and v of G. In this case, the coloring c is called a *rainbow coloring* of G. If k colors are used, then c is a *rainbow k-coloring*. The *rainbow connection number* of G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. A *rainbow $u-v$ geodesic* in G is a *rainbow $u-v$ path* of length $d(u, v)$, where $d(u, v)$ is the distance between u and v. The graph G is *strongly rainbow connected* if there exists a rainbow $u - v$ geodesic for any two vertices u and v in G. In this case, the coloring c is called a *strong rainbow coloring* of G. Similarly, we define the *strong rainbow connection number* of a connected graph G, denoted by $src(G)$, as the smallest number of colors that are needed in order to make G strong rainbow connected. Clearly, we have $diam(G) \leq rc(G) \leq scr(G) \leq m$, where $diam(G)$ denotes the diameter of G and m is the number of edges of G. Moreover, it is easy to verify that $src(G) = rc(G) = 1$ if and only if G is a complete graph, that $rc(G) = 2$ if and only if $src(G) = 2$, and that $rc(G) = n-1$ if and only if G is a tree. The concepts of rainbow connectivity and strong rainbow connectivity were first introduced by Chartrand et al. in [5] as a means of strengthening the connectivity. Subsequent to this paper, the problem has received attention by several people and the complexity as well as upper bounds for the rainbow connection number have been studied.

In [3], Caro et al. conjectured that computing $rc(G)$ is an NP-Hard problem, as well as that even deciding whether a graph has $rc(G) = 2$ is NP-Complete. In [4], Chakraborty et al. confirmed this conjecture. In [1], the complexity of computing $rc(G)$ and $src(G)$ was studied further. It was shown that given any natural number $k \geq 3$ and a graph G, it is NP-hard to determine whether $rc(G) \leq k$. Moreover, for $src(G)$, it was shown that given any natural number $k \geq 3$ and a graph G, determining whether $src(G) \leq k$ is NP-hard even when G is bipartite. In this paper, we will point out that the problems in [1] are, in fact, NP-Complete. Though for a general graph G it is NP-Complete to decide whether $rc(G) = 2$ [4], we show that the problem becomes easy when G is a bipartite graph. Moreover, it is NP-Complete to decide whether a given edge-colored (with an unbound number of colors) graph is rainbow connected [4]. We will prove that it is still

NP-Complete even when the edge-colored graph is bipartite.

2   Main results

At first, we restate several results in [4] and [1].

Lemma 2.1. *([4]) Given a graph G, deciding if $rc(G) = 2$ is NP-Complete. In particular, computing $rc(G)$ is NP-Hard.*

Lemma 2.2. *([1]) For every $k \geq 3$, deciding whether $rc(G) \leq k$ is NP-Hard.*

Lemma 2.3. *([1]) Deciding whether the rainbow connection number of a graph is at most 3 is NP-Hard even when the graph G is bipartite.*

Lemma 2.4. *([1]) For every $k \geq 3$, deciding whether $src(G) \leq k$ is NP-Hard even when G is bipartite.*

We will show that "NP-hard" in the above results can be replaced by "NP-Complete" if k is any fixed integer. It suffices to show that these problems belong to the class NP for any fixed k. In fact, from the proofs in [1], for the problems in Lemmas 2.2 and 2.4, "For every $k \geq 3$" can be replaced by "For any fixed $k \geq 3$".

Theorem 2.1. *For any fixed $k \geq 2$, given a graph G, deciding whether $rc(G) \leq k$ is NP-Complete.*

Proof. By Lemmas 2.1 and 2.2, it will suffice to show that the problem in Lemma 2.2 is in NP. Therefore, if given any instance of the problem whose answer is 'yes', namely a graph G with $rc(G) \leq k$, we want to show that there is a certificate validating this fact which can be checked in polynomial time.

Obviously, a rainbow k-coloring of G means that $rc(G) \leq k$. For checking a rainbow k-coloring, we need only check whether k colors are used and for any two vertices u and v of G, whether there exists a rainbow $u - v$ path. Notice that for two vertices u, v, there are at most n^{l-1} $u - v$ paths of length l, since if let $P = ut_1t_2 \cdots t_{l-1}v$, there are less than n choices for each t_i $(i \in \{1, 2, \ldots, l-1\})$. Therefore, G contains at most $\Sigma_{l=1}^{k} n^{l-1} \leq kn^{k-1} \leq n^k$ $u - v$ paths of length no more than k. Then check these paths in turn until find one path whose edges have distinct colors or no such paths at all. It follows that the time used for checking is at most $O(n^k \cdot n \cdot n^2) = O(n^{k+3})$. Since k is a fixed integer, we conclude that the certificate, namely a rainbow k-coloring of G, can be checked in polynomial time. The proof is now complete. ∎

The next theorem can be obtained similarly.

Theorem 2.2. *For any fixed $k \geq 2$, given a graph G, deciding whether $src(G) \leq k$ is NP-Complete.*

Proof. Since $rc(G) = 2$ if and only if $src(G) = 2$, by Lemmas 2.1 and 2.4, it will suffice to show that the problem in Lemma 2.4 is in NP.

From the proof of Theorem 2.1, it is clear that for any two vertices u and v of G, the existence of a $u - v$ path of length l ($\leq k$) can be decided in time $O(n^{l-1})$. Therefore, if we check each integer $l \leq k$ in turn, we can either find an integer l such that there is a $u - v$ path of length l but no $u - v$ path of length less than l, or conclude that there is no $u - v$ path of length at most k. In the former case, the integer l is exactly the distance $d(u, v)$ between u and v and then check the colors of edges of each $u - v$ path of length $d(u, v)$ in turn. Similarly to the proof of Theorem 2.1, we can obtain that the certificate, namely a strong rainbow k-coloring of G, can be checked in polynomial time. The proof is complete. ∎

We know that given a graph G, deciding if $rc(G) = 2$ is NP-Complete. Surprisingly, if G is a bipartite graph, the problem turns out to be easy. Before giving the proof, we first introduce the following result of [5].

Lemma 2.5. *([5]) For integers s and t with $2 \leq s \leq t$,*

$$rc(K_{s,t}) = min\{\lceil \sqrt[s]{t} \rceil, \, 4\}.$$

Theorem 2.3. *For a bipartite graph G, deciding whether $rc(G) = 2$ can be solved in polynomial time.*

Proof. Obviously if G is not a complete bipartite graph, there must exist two nonadjacent vertices x and y in the different parts of G. But then the distance $d(x, y)$ must be at least 3. We know that $d(x, y) \leq diam(G) \leq rc(G)$. It follows that $rc(G) \neq 2$. Therefore, only when G is a complete bipartite graph $K_{s,t}$ ($s \leq t$), it is possible that $rc(G) = 2$. If $s = 1$, then G is a star and $rc(G) = t$. Otherwise by Lemma 2.5, $rc(G) = min\{\lceil \sqrt[s]{t} \rceil, \, 4\}$. One needs only to check if $1 < t \leq 2^s$, which can be done by simple computation and comparison. Moreover, it is clear that checking whether G is a complete bipartite graph can be done in polynomial time. The proof is complete. ∎

Then by Lemma 2.3, the following result is immediate.

Corollary 2.1. *Given a bipartite graph G, deciding if $rc(G) = 3$ is NP-Complete.*

As shown in the proof of Theorem 2.1, given an edge-coloring of a graph, if the number of colors is constant, then we can verify whether the colored graph is rainbow connected in polynomial time. However, in [4], chakraborty et al. showed that if the coloring is arbitrary, the problem becomes NP-Complete.

Lemma 2.6. *([4]) The following problem is NP-Complete: Given an edge-colored graph G, check whether the given coloring makes G rainbow connected.*

Now we prove that even when G is bipartite, the problem is still NP-Complete.

Theorem 2.4. *Given an edge-colored bipartite graph G, checking whether the given coloring makes G rainbow connected is NP-Complete.*

Proof. By Lemma 2.6, it will suffice by showing a polynomial reduction from the problem in Lemma 2.6.

Given a graph $G = (V, E)$ and an edge-coloring c of G, we will construct an edge-colored bipartite graph G' such that G is rainbow connected if and only if G' is rainbow connected.

Now for each edge $e \in E(G)$, subdivide e by a new vertex v_e. The obtained graph is exactly G' and (X, Y) is a bipartition of G', where $X = V(G)$ and $Y = \{v_e \mid e \in E(G)\}$. Then the edge-coloring c' of G' is defined by for each edge $e = v_i v_j \in E(G)$ $(i \leq j)$, $c'(v_i v_e) = c(e)$ and $c'(v_j v_e) = l_e$, where l_e is a new color and different from the colors used in c and if $e \neq e'$, then $l_e \neq l_{e'}$.

If c' is a rainbow coloring of G', then any two vertices u and v are connected by a rainbow path $P'_{u,v}$, including every pair of vertices in $X = V(G)$. Clearly, by contracting edges which are assigned new colors, $P'_{u,v}$ can be converted to a rainbow path $P_{u,v}$ of G (with respect to c), where $u, v \in V(G)$. It follows that the coloring c makes G rainbow connected.

To prove the other direction, assume that for every two vertices v_t and $v_{t'}$ of G, there always exists a rainbow path $P_{v_t v_{t'}} = v_t v_{t_1} v_{t_2} \ldots v_{t'}$. Now for each pair $(v_t, v_{t'})$ of vertices in $V(G')$, if $v_t, v_{t'} \in X = V(G)$, then $P'_{v_t v_{t'}} = v_t v_{e_{m_1}} v_{t_1} v_{e_{m_2}} v_{t_2} \ldots v_{e_{m_j}} v_{t'}$ is a rainbow path in G', where the vertex $v_{e_{m_i}}$ subdivides the edge $e_{m_i} = v_{t_{i-1}} v_{t_i}$ of G, $i \in \{1, \ldots, j\}$ (when $i = 1$, the edge is $v_t v_{t_1}$ and when $i = j$, the edge is $v_{t_{j-1}} v_{t'}$). If $v_t, v_{t'} \in Y$, then there exist two edges $e_1 = v_{i_1} v_{j_1}$ and $e_2 = v_{i_2} v_{j_2}$ $(i_1 \leq j_1$ and $i_2 \leq j_2)$ such that in G' v_t and $v_{t'}$ subdivide e_1 and e_2, respectively. Since $v_{j_1}, v_{j_2} \in X = V(G)$, we can find a rainbow path $P'_{v_{j_1} v_{j_2}}$ in G' which can be converted to a rainbow $v_t - v_{t'}$ path $P'_{v_t v_{t'}} = v_t v_{j_1} P'_{v_{j_1} v_{j_2}} v_{j_2} v_{t'}$.

The proof of the case that $v_t \in X$ and $v_{t'} \in Y$ is similar. Therefore G' is rainbow connected with respect to c'. The proof is complete. ∎

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